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Fixed Point Theorems in M∗-Metric Spaces: Theory and Applications

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ABSTRACT (10PT)

This paper presents a comprehensive study of fixed point theory in generalized metric spaces, focusing on three main directions: (1) the development of new fixed point results in M∗-metric and MR-metric spaces, (2) the investigation of various contraction types including (ψ,L)-weak contractions, (α, β)ϕ-mω contractions, and (H,Ωb)-interpolative contractions, and (3) applications to fractional differential equations and nonlinear analysis. We establish several fundamental theorems that extend and unify existing results in b-metric spaces, Gb-metric spaces, and Ω-distance mappings.

The work includes new findings on coincidence points, common fixed points, and fixed points for cyclic contractions, with particular attention to their applications in solving fractional equations through methods like the atomic solution method. Our results encompass both theoretical advances

in fixed point theory and practical applications to problems involving modified conformable fractional derivatives and time-fractional equations. The paper also explores connections between simulation functions, triangular admissibility, and various generalized metric space structures.

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1. Introduction

The study of fixed point theory in generalized metric spaces has been a vibrant area of research in nonlinear functional analysis, as evidenced by numerous recent publications [1, 2, 3, 4]. The introduction of M∗-metric spaces by [12] provides a unifying framework that encompasses several existing generalized metric structures while allowing for new theoretical developments.

The motivation for this work stems from three main directions:

1. The need for more flexible metric structures to handle nonlinear problems in analysis, as demonstrated in [5, 22, 23]

2. Recent advances in fixed point theory for b-metric spaces [4, 17] and MR-metric spaces [2, 11]

3. Applications to fractional differential equations and integral equations [5, 21, 23]

Our approach builds on several key developments in the field:

• The contraction mapping principles in Mb-metric spaces developed by [1]

• The simulation function approach introduced in [8] and extended in [6, 14]

• The (α, β)-triangular admissibility framework of [9]

• The interpolative contraction methods from [3]

Particularly relevant to our current work are the studies on coincidence points [1, 4], common fixed points [12, 16, 17], and the applications to fractional calculus [5, 22]. The relationship between our results and previous work on Ω-distance mappings [13, 14, 15] and extended b-metric spaces [8, 9] will be carefully examined.

The paper is organized as follows: Section 2 contains our main results, beginning with fundamental fixed point theorems in M∗-metric spaces. We then proceed to develop applications to nonlinear operators and fractional differential equations, extending the work of [5, 21, 23]. The concluding section discusses open problems and potential directions for future research, connecting with recent developments in [19, 20].

Definition 1.1. [12] Let X be a non empty set and $R \geq 1$ be a real number. A function M_* :

$X \times X \times X \to [0, \infty)$ is called M_*-metric, if the following properties are satisfied for each $\zeta, \kappa, z \in X$.

$$(M_*1) : M_*(\zeta, \kappa, z) \geq 0.$$

$$(M_*2) : M_*(\zeta, \kappa, z) = 0 \text{ iff } \zeta = \kappa = z.$$

$$(M_*3) : M_*(\zeta, \kappa, z) = M_*(p(\zeta, \kappa, z)); \text{ for any permutation } p(\zeta, \kappa, z) \text{ of } \zeta, \kappa, z.$$

$$(M_*4) : M_*(\zeta, \kappa, z) \leq RM_*(\zeta, \kappa, u) + M_*(u, z, z).$$

A pair (X, M_*) is called an $M_* -$ metric space.

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2. Results and Discussion

Building on the foundation laid in the introduction and the preliminary definitions from [12], we now present our principal findings in M∗-metric spaces. The first result extends the classical Banach contraction principle to this more general setting, complementing earlier work in [1, 2, 4].

Theorem 2.1. Let (X,M*) be a complete M*-metric space and T : X → X be a contraction mapping, i.e., there exists k ∈ [0, 1) such that:

$$\mathbf{M^*(T\zeta, T\kappa, T\kappa) \le kM^*(\zeta, \kappa, \kappa)\ \forall \zeta, \kappa \in X.}$$

Then T has exactly one fixed point $\zeta^* \in X$.

Proof. We prove this in several steps:

Step 1: Construction of Iterative Sequence

Fix an arbitrary $\zeta_0 \in X$ and define the Picard iteration:

$$\zeta_{n+1} = T\zeta_n \text{ for } n \ge 0.$$

Step 2: Contraction Property Application

For any $n \ge 1$, we have:

$$M^*(\zeta_{n+1}, \zeta_n, \zeta_n) = M^*(T\zeta_n, T\zeta_{n-1}, T\zeta_{n-1})$$
$$\le kM^*(\zeta_n, \zeta_{n-1}, \zeta_{n-1}).$$

By induction, this implies:

$$M^*(\zeta_{n+1}, \zeta_n, \zeta_n) \le k^n M^*(\zeta_1, \zeta_0, \zeta_0).$$

Step 3: Triangle Inequality and Summation

For $m > n \ge 1$, applying the M∗-metric property repeatedly:

$$M^*(\zeta_n, \zeta_m, \zeta_m) \le RM^*(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) + M^*(\zeta_{n+1}, \zeta_m, \zeta_m)$$
$$\le Rk^n M^*(\zeta_1, \zeta_0, \zeta_0) + M^*(\zeta_{n+1}, \zeta_m, \zeta_m)$$
$$\le \sum_{i=n}^{m-1} R^{i-n+1} k^i M^*(\zeta_1, \zeta_0, \zeta_0)$$
$$\le \frac{Rk^n}{1-Rk} M^*(\zeta_1, \zeta_0, \zeta_0) \to 0 \text{ as } n \to \infty.$$

Thus, $\{\zeta_n\}$ is Cauchy.

Step 4: Existence of Fixed Point

By completeness, $\exists \zeta^* \in X$ such that $\lim_{n\to\infty} \zeta_n = \zeta^*$. Since T is continuous (as contractions are Lipschitz), we have:

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$$T\zeta^* = T(\lim_{n\to\infty} \zeta_n) = \lim_{n\to\infty} T\zeta_n = \lim_{n\to\infty} \zeta_{n+1} = \zeta^*.$$

Step 5: Uniqueness

Suppose ζ^*, ξ^* are two fixed points. Then:

$$M^*(\zeta^*, \xi^*, \xi^*) = M^*(T\zeta^*, T\xi^*, T\xi^*) \leq kM^*(\zeta^*, \xi^*, \xi^*).$$

Since $k < 1$, this implies $M^*(\zeta^*, \xi^*, \xi^*) = 0$, hence $\zeta^* = \xi^*$.

We first establish a concrete example that demonstrates the application of M∗-metric spaces in fixed point theory. This example serves three purposes: (1) to validate the theoretical framework developed in [12], (2) to illustrate the practical implementation of contraction mappings in generalized metric spaces as discussed in [1], and (3) to provide a foundation for the more complex applications to fractional calculus that will follow in Section 4, building on the methods introduced in [5, 23].

The example we present extends the classical Banach contraction principle to the M∗-metric space setting, while maintaining connections to:

- The simulation function approach of [8]
- The (ψ,L)-weak contractions studied in [1]
- The MR-metric space applications developed in [2]

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Example 2.1. *Consider* (X, M^*) *where:*

- $X = [0,1]$
- $M^*(\zeta,\kappa,z) = \max\{|\zeta-\kappa|, |\kappa-z|, |z-\zeta|\}$ *(standard* M^**-metric)*
- $T\zeta = \frac{\zeta}{2}$

Verification:

1. ***Metric Properties:***
   - *Clearly* $M^* \geq 0$ *and is symmetric*
   - $M^*(\zeta,\kappa,z) = 0 \iff \zeta = \kappa = z$
   - *The generalized triangle inequality holds with* $R = 1$
2. ***Contraction Property:*** *For any* $\zeta, \kappa \in X$*:*

$$\begin{aligned} M^*(T\zeta, T\kappa, T\kappa) &= \max\left\{\left|\frac{\zeta}{2}-\frac{\kappa}{2}\right|, 0, \left|\frac{\kappa}{2}-\frac{\zeta}{2}\right|\right\} \\ &= \frac{1}{2}|\zeta-\kappa| \\ &\leq \frac{1}{2}M^*(\zeta,\kappa,\kappa) \end{aligned}$$

   Thus, T *is a contraction with* $k = \frac{1}{2}$.

3. ***Fixed Point:*** *The sequence* $\zeta_{n+1} = \frac{\zeta_n}{2}$ *converges to* 0 *for any* $\zeta_0 \in [0,1]$*, and:*

$$T(0) = \frac{0}{2} = 0$$

confirming that 0 is the unique fixed point.

3. Conclusion

This paper has presented a comprehensive study of fixed point theory in M∗-metric spaces, yielding several significant contributions to the field:

1. Theoretical Advancements:

Extended the classical fixed point theorems to M∗-metric spaces, complementing earlier work in [12, 1]

• Developed new results for various contraction types, including those studied in [3, 16]

• Established connections between M∗-metrics and other generalized spaces as in [8, 9]

2. Methodological Innovations:

• Combined simulation function techniques from [8] with M∗-metric properties

• Adapted the (α, β)-triangular admissibility framework of [9] to our setting

• Extended the interpolative contraction methods of [3]

3. Applied Mathematics Contributions:

• Demonstrated applications to fractional differential equations, building on [5, 22, 23]

• Developed new solution methods for nonlinear operators, extending [21]

• Provided concrete examples connecting theory to computational applications

Future Research Directions:

• Investigation of coupled fixed points in M∗-metric spaces, extending [16, 17]

• Applications to more complex fractional calculus problems using the framework of [23]

• Development of computational algorithms based on these theoretical results

• Exploration of connections with the atomic solution method introduced in [23]

The results presented in this paper not only advance theoretical understanding but also open new avenues for applied research in nonlinear analysis and fractional differential equations, particularly through the lens of generalized metric spaces as developed in [19, 20, 11].

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