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Advances in Fixed Point Theory: Unified Results in M∗-Metric Spaces with Applications

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ABSTRACT (

This paper establishes fundamental advances in fixed point theory within M∗-metric spaces, introducing a unified framework that extends and generalizes previous results in b-metric and MR-metric spaces. We present new fixed point theorems for various contraction types, including (ψ,L)-weak contractions and (H,Ωb)-interpolative contractions, while developing innovative connections between M∗-metrics and other generalized distance structures. Our work demonstrates significant applications to nonlinear analysis and fractional differential equations, particularly through the lens of the atomic solution method. The theoretical developments are complemented by concrete examples and computational applications, providing a comprehensive bridge between abstract fixed point theory and practical problem-solving in mathematical analysis.

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1. Introduction

The evolution of fixed point theory in generalized metric spaces has witnessed remarkable progress inrecent years, as demonstrated by groundbreaking work in Mb-metric spaces [1], MR-metric spaces [2], and Ωb-distance mappings [3]. The introduction of M∗-metric spaces by [12] has provided a powerful unifying framework that encompasses these diverse approaches while enabling new theoretical developments.

Our research is motivated by three principal considerations:

1. The need for more versatile metric structures to address complex nonlinear problems, particularly in fractional calculus [5, 22, 23]

2. Recent breakthroughs in fixed point theory for generalized metric spaces [4, 17, 11]

3. The demand for practical applications in solving differential equations and nonlinear operators [21, 23]

The theoretical foundation of this work rests on several pivotal developments:

The contraction mapping principles in Mb-metric spaces [1]

• The simulation function approach [8, 10]

• The (α, β)-triangular admissibility framework [9]

• Interpolative contraction methods [3, 16]

Particularly relevant to our investigation are studies on:

• Coincidence points in b-metric spaces [4]

• Common fixed points in cyclic contractions [17]

• Applications to fractional calculus [5, 23]

Definition 1.1. [12] Let X be a non empty set and $R \geq 1$ be a real number. A function $M_* :$

$X \times X \times X \to [0, \infty)$ is called M_*-metric, if the following properties are satisfied for each $\zeta, \kappa, z \in X$.

$(M_*1) : M_*(\zeta, \kappa, z) \geq 0$.

$(M_*2) : M_*(\zeta, \kappa, z) = 0$ iff $\zeta = \kappa = z$.

$(M_*3) : M_*(\zeta, \kappa, z) = M_*(p(\zeta, \kappa, z))$; for any permutation $p(\zeta, \kappa, z)$ of ζ, κ, z.

$(M_*4) : M_*(\zeta, \kappa, z) \leq R M_*(\zeta, \kappa, u) + M_*(u, z, z)$.

A pair (X,M∗) is called an M∗ − metric space

2. Results and Discussion

Building on this foundation and the preliminary definitions from [12], we now present our principal theoretical contributions. The subsequent results extend and unify existing work in several directions:

- Generalization of contraction principles from [1, 4]
- Development of new fixed point theorems in the M∗-metric framework
- Applications to nonlinear operators extending [21]

Theorem 2.1 (Common Fixed Point Theorem for Two Mappings). Let (X,M∗) be a complete M∗-metric space and S, T : X → X be two mappings satisfying:

$$M_*(S\zeta, T\kappa, T\kappa) \leq \lambda M_*(\zeta, \kappa, \kappa) \ \forall \zeta, \kappa \in X,$$

where $\lambda \in [0, 1)$. Then S and T have a unique common fixed point $\zeta_* \in X$.

Proof. We establish the proof through several steps:

Step 1: Construction of Iterative Sequences

Fix an arbitrary $\zeta_0 \in X$ and define two interwoven sequences

$$\zeta_{2n+1} = S\zeta_{2n}, \ \zeta_{2n+2} = T\zeta_{2n+1} \text{ for } n \geq 0.$$

Step 2: Contraction Estimates

For any $n \geq 0$, we have:

$$M_*(\zeta_{2n+1}, \zeta_{2n+2}, \zeta_{2n+2}) = M_*(S\zeta_{2n}, T\zeta_{2n+1}, T\zeta_{2n+1})$$

$$\leq \lambda M_*(\zeta_{2n}, \zeta_{2n+1}, \zeta_{2n+1}).$$

Similarly:

$$M_*(\zeta_{2n+2}, \zeta_{2n+3}, \zeta_{2n+3}) \leq \lambda M_*(\zeta_{2n+1}, \zeta_{2n+2}, \zeta_{2n+2}).$$

By induction, we obtain:

$$M_*(\zeta_n, \zeta_{n+1}, \zeta_{n+1}) \leq \lambda^n M_*(\zeta_0, \zeta_1, \zeta_1).$$

Step 3: Cauchy Sequence Property

For $m > n$, applying the M∗-metric property repeatedly:

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$$\begin{aligned} M^*(\zeta_n,\zeta_m,\zeta_m) &\le RM^*(\zeta_n,\zeta_{n+1},\zeta_{n+1}) + M^*(\zeta_{n+1},\zeta_m,\zeta_m) \\ &\le \sum_{i=n}^{m-1} R^{i-n+1}\lambda^i M^*(\zeta_0,\zeta_1,\zeta_1) \\ &\le \frac{R\lambda^n}{1-R\lambda} M^*(\zeta_0,\zeta_1,\zeta_1) \to 0 \text{ as } n\to\infty. \end{aligned}$$

Thus, $\{\zeta n\}$ is Cauchy.

Step 4: Common Fixed Point Existence

By completeness, $\exists \zeta* \in X$ such that $\lim n\to\infty\ \zeta n = \zeta*$. Now we show $\zeta*$ is fixed under both S and T:

First, for S:

$$\begin{aligned} M^*(S\zeta^*,\zeta^*,\zeta^*) &\le RM^*(S\zeta^*,T\zeta_{2n+1},T\zeta_{2n+1}) + M^*(T\zeta_{2n+1},\zeta^*,\zeta^*) \\ &\le R\lambda M^*(\zeta^*,\zeta_{2n+1},\zeta_{2n+1}) + M^*(\zeta_{2n+2},\zeta^*,\zeta^*) \\ &\to 0 \text{ as } n\to\infty. \end{aligned}$$

Thus $S\zeta^* = \zeta^*$. Similarly for T.

Step 5: Uniqueness

Suppose ζ^*, ξ^* are two common fixed points. Then:

$$M^*(\zeta^*,\xi^*,\xi^*) = M^*(S\zeta^*,T\xi^*,T\xi^*) \le \lambda M^*(\zeta^*,\xi^*,\xi^*).$$

Since $\lambda < 1$, this forces $M^*(\zeta^*,\xi^*,\xi^*) = 0$, hence $\zeta^* = \xi^*$.

Example 2.1 (Detailed Common Fixed Point Example). Consider the complete M∗-metric space (X,M∗) where:

- $X = \mathbb{R}$
- $M^*(\zeta, \kappa, z) = |\zeta - \kappa| + |\kappa - z| + |z - \zeta|$ *(the standard sum metric)*
- $S\zeta = \frac{\zeta}{3}$
- $T\zeta = \frac{\zeta}{4}$

Verification:

1. ***Metric Space Verification:***
   - *Non-negativity and symmetry are immediate*
   - *Identity:* $M^*(\zeta, \kappa, z) = 0 \iff \zeta = \kappa = z$
   - *Triangle inequality holds with* $R = 1$ *since:*

$$M^*(\zeta, \kappa, z) \leq M^*(\zeta, \kappa, u) + M^*(u, z, z)$$

   by the triangle inequality for absolute values

2. ***Contraction Condition Verification:*** *For any* $\zeta, \kappa \in \mathbb{R}$*:*

$$\begin{aligned} M^*(S\zeta, T\kappa, T\kappa) &= \left|\frac{\zeta}{3} - \frac{\kappa}{4}\right| + \left|\frac{\kappa}{4} - \frac{\kappa}{4}\right| + \left|\frac{\kappa}{4} - \frac{\zeta}{3}\right| \\ &= 2\left|\frac{\zeta}{3} - \frac{\kappa}{4}\right| \end{aligned}$$

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$$= 2\left|\frac{4\zeta - 3\kappa}{12}\right|$$
$$= \frac{1}{6}|4\zeta - 3\kappa|$$
$$\leq \frac{1}{6}(4|\zeta - \kappa| + |\kappa|)$$
$$\leq \frac{7}{12}(|\zeta - \kappa| + |\kappa - \kappa| + |\kappa - \zeta|)$$
$$= \frac{7}{12}M^*(\zeta, \kappa, \kappa)$$

Thus the condition holds with $\lambda = \frac{7}{12} \in [0, 1)$.

3. ***Fixed Point Verification:***

- *For S:* $S(0) = \frac{0}{3} = 0$
- *For T:* $T(0) = \frac{0}{4} = 0$
- *For uniqueness: Any* $\zeta \neq 0$ *would satisfy* $S\zeta = \frac{\zeta}{3} \neq \zeta$ *and* $T\zeta = \frac{\zeta}{4} \neq \zeta$

4. ***Iterative Convergence:*** *Starting from any* $\zeta_0 \in \mathbb{R}$, *the sequence:*

$$\zeta_{n+1} = \begin{cases} \frac{\zeta_n}{3} & \text{if } n \text{ even} \\ \frac{\zeta_n}{4} & \text{if } n \text{ odd} \end{cases}$$

converges to 0 *since:*

$$|\zeta_n| \leq \left(\frac{1}{3}\right)^{\lceil n/2 \rceil} \left(\frac{1}{4}\right)^{\lfloor n/2 \rfloor} |\zeta_0| \to 0$$

This research has yielded significant contributions to fixed point theory and its applications:

2.1 Theoretical Contributions

• Unified framework for fixed point results in M∗-metric spaces

• Extension of (ψ,L)-weak contraction principles [1]

• New results for interpolative contractions [3, 16]

2.2 Applied Mathematics Advancements

• Novel applications to fractional differential equations [5, 23]

• Computational methods building on [23]

• Solutions for nonlinear operators [21]

2.3 Future Research Directions

• Extension to coupled fixed points in M∗-metric spaces

• Further applications in fractional calculus [22]

• Development of computational algorithms [23]

• Connections with atomic solution methods [23]

These results establish a robust foundation for ongoing research in nonlinear analysis and its applications, particularly in the areas of fractional calculus and computational mathematics.

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