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Advances in Fixed Point Theory: ´Ciri´c-Type Contractions in M∗-Metric Spaces

Fatima Hussien Hussien [a,1,*], Basma Zaydoun [b,2],

[a] Phd, Department of Mathematics, College of Science, Alanbar University, Iraq-Alanbar
[b] Prof., Department of Mathematics, College of Science, Alanbar University, Iraq-Alanbar

* Corresponding Author

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ABSTRACT (10PT)

This paper presents significant developments in fixed point theory within M∗-metric spaces, focusing on ´Ciri´c-type contractions and their applications. We establish new fixed point theorems that extend and generalize previous results in Mb-metric and MR-metric spaces. The work includes comprehensive analysis of contraction conditions, detailed examples, and applications to nonlinear problems.

Our results unify various approaches from generalized metric space theory while providing novel insights into the structure of M∗-metric spaces. The theoretical developments are complemented by practical applications, demonstrating the utility of these concepts in solving complex mathematical problems, particularly in the context of fractional calculus and nonlinear operators.

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1. Introduction

The study of fixed point theory in generalized metric spaces has witnessed remarkable progress in recent years, with groundbreaking work in Mb-metric spaces [1], MR-metric spaces [2], and various extended metric structures [3, 6, 7]. The development of M∗-metric spaces by [12] has provided a comprehensive framework that encompasses and extends these previous approaches.

Our research is motivated by three principal considerations:

1. The need for more sophisticated contraction conditions in fixed point theory, as demonstrated in [4, 17]

2. Recent advances in ´Ciri´c-type fixed point theorems [1, 11]

3. Applications to nonlinear problems and fractional calculus [5, 23]

The theoretical foundation of this work builds upon several pivotal developments:

- The (ψ,L)-weak contraction principles in Mb-metric spaces [1]
- The simulation function approach developed in [8, 10]
- The (α, β)-triangular admissibility framework [9]
- Interpolative contraction methods [3, 16]

Particularly relevant to our current investigation are studies on:

- Fixed points in generalized metric spaces [4]
- Common fixed points for cyclic contractions [17]
- Applications to fractional differential equations [5, 23]

Definition 1.1. [12] Let X be a non empty set and $R \geq 1$ be a real number. A function M∗ :

$X\times X\times X \to [0,\infty)$ is called M∗−metric, if the following properties are satisfied for each $\zeta, \kappa, z \in X$.

(M∗1) : $M_*(\zeta, \kappa, z) \geq 0$.

(M∗2) : $M*(\zeta, \kappa, z) = 0$ iff $\zeta = \kappa = z$.

(M∗3) : $M*(\zeta, \kappa, z) = M*(p(\zeta, \kappa, z))$; for any permutation $p(\zeta, \kappa, z)$ of ζ, κ, z.

(M∗4) : $M*(\zeta, \kappa, z) \leq RM*(\zeta, \kappa, u) + M*(u, z, z)$.

A pair $(X, M*)$ is called an $M* -$ metric space.

2. Results and Discussion

Building on these foundations and the preliminary definitions of M∗-metric spaces [12], we now present our principal theoretical contributions. The subsequent results extend the classical ´Ciri´c fixed point theorem in several important directions:

- New fixed point theorems for ´Ciri´c-type contractions in M∗-metric spaces
- Applications to nonlinear operators extending [21]
- Connections with other generalized metric space structures [6, 7]

Theorem 2.1 (´Ciri´c-Type Fixed Point Theorem). Let $(X, M*)$ be a complete M∗-metric space and $T : X \to X$ be a mapping satisfying:

$$M^*(T\zeta, T\kappa, T\kappa) \leq \lambda \max \left\{ \begin{array}{l} M^*(\zeta, \kappa, \kappa), \\ M^*(\zeta, T\zeta, T\zeta), \\ M^*(\kappa, T\kappa, T\kappa) \end{array} \right\}$$

for some $\lambda \in [0, 1)$ *and all* $\zeta, \kappa \in X$. *Then* T *has a unique fixed point* $\zeta^* \in X$.

Proof. We establish the proof through several steps:

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Step 1: Iterative Sequence Construction

Fix an arbitrary $\zeta_0 \in X$ and define the Picard iteration:

$$\zeta_{n+1} = T\zeta_n \quad \text{for } n \geq 0.$$

Step 2: Establishing the Key Inequality

Applying the Ćirić condition to consecutive terms:

$$\begin{aligned} M^*(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}) &= M^*(T\zeta_n, T\zeta_{n+1}, T\zeta_{n+1}) \\ &\leq \lambda \max \left\{ \begin{array}{l} M^*(\zeta_n, \zeta_{n+1}, \zeta_{n+1}), \\ M^*(\zeta_n, T\zeta_n, T\zeta_n), \\ M^*(\zeta_{n+1}, T\zeta_{n+1}, T\zeta_{n+1}) \end{array} \right\} \\ &= \lambda \max \left\{ M^*(\zeta_n, \zeta_{n+1}, \zeta_{n+1}), M^*(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}) \right\}. \end{aligned}$$

Case Analysis:

- If the maximum is $M^*(\zeta_n, \zeta_{n+1}, \zeta_{n+1})$:

$$M^*(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}) \leq \lambda M^*(\zeta_n, \zeta_{n+1}, \zeta_{n+1})$$

- If the maximum is $M^*(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2})$:

$$M^*(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}) \leq \lambda M^*(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2})$$

  which implies $M^*(\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+2}) = 0$ since $\lambda < 1$.

Step 3: Contraction-Type Behavior

In both cases, we obtain:

$$M^*(\zeta_{n+1},\zeta_{n+2},\zeta_{n+2}) \leq \lambda M^*(\zeta_n,\zeta_{n+1},\zeta_{n+1}).$$

By induction:

$$M^*(\zeta_n,\zeta_{n+1},\zeta_{n+1}) \leq \lambda^n M^*(\zeta_0,\zeta_1,\zeta_1).$$

Step 4: Cauchy Sequence Verification

For $m > n \geq 1$, using the M^*-metric property:

$$\begin{aligned} M^*(\zeta_n,\zeta_m,\zeta_m) &\leq RM^*(\zeta_n,\zeta_{n+1},\zeta_{n+1}) + M^*(\zeta_{n+1},\zeta_m,\zeta_m) \\ &\leq \sum_{i=n}^{m-1} R^{i-n+1}\lambda^i M^*(\zeta_0,\zeta_1,\zeta_1) \\ &\leq \frac{R\lambda^n}{1-R\lambda} M^*(\zeta_0,\zeta_1,\zeta_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $\{\zeta_n\}$ is Cauchy.

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Step 5: Fixed Point Existence

By completeness, $\exists \zeta^* \in X$ such that $\lim_{n\to\infty} \zeta_n = \zeta^*$. Now verify ζ^* is fixed:

$$\begin{aligned} M^*(T\zeta^*,\zeta^*,\zeta^*) &\leq RM^*(T\zeta^*,T\zeta_n,T\zeta_n) + M^*(T\zeta_n,\zeta^*,\zeta^*) \\ &\leq R\lambda \max \left\{ \begin{array}{l} M^*(\zeta^*,\zeta_n,\zeta_n), \\ M^*(\zeta^*,T\zeta^*,T\zeta^*), \\ M^*(\zeta_n,T\zeta_n,T\zeta_n) \end{array} \right\} \\ &\quad + M^*(\zeta_{n+1},\zeta^*,\zeta^*) \\ &\to R\lambda M^*(\zeta^*,T\zeta^*,T\zeta^*) \text{ as } n\to\infty. \end{aligned}$$

Since $R\lambda < 1$, this implies $M^*(T\zeta^*,\zeta^*,\zeta^*) = 0$.

Step 6: Uniqueness

Suppose ζ^*, ξ^* are two fixed points. Then:

$$M^*(\zeta^*,\xi^*,\xi^*) = M^*(T\zeta^*,T\xi^*,T\xi^*) \leq \lambda \max \left\{ \begin{array}{l} M^*(\zeta^*,\xi^*,\xi^*), \\ M^*(\zeta^*,T\zeta^*,T\zeta^*), \\ M^*(\xi^*,T\xi^*,T\xi^*) \end{array} \right\} = \lambda M^*(\zeta^*,\xi^*,\xi^*).$$

Since $\lambda < 1$, we must have $M^*(\zeta^*,\xi^*,\xi^*) = 0$. □

Example 2.1 (Detailed Ćirić-Type Example)**.** *Consider the M^*-metric space (X, M^*) where:*

- $X = [0,1]$
- $M^*(\zeta,\kappa,z) = |\zeta-\kappa| + |\kappa - z|$
- *Mapping* $T\zeta = \frac{\zeta^2}{2}$

Verification:

1. ***Metric Space Verification:***
    - *Non-negativity and symmetry are clear*
    - *Identity:* $M^*(\zeta,\kappa,z) = 0 \iff \zeta=\kappa=z$
    - *Triangle inequality holds with* $R=1$*:*

$$|\zeta-\kappa| + |\kappa - z| \leq (|\zeta-u| + |u-\kappa|) + (|\kappa-u| + |u-z|)$$

2. ***Ćirić Condition Verification:*** *We need to show:*

$$M^*(T\zeta,T\kappa,T\kappa) \leq \frac{1}{2}\max\{M^*(\zeta,\kappa,\kappa), M^*(\zeta,T\zeta,T\zeta), M^*(\kappa,T\kappa,T\kappa)\}$$

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Compute:

$$\begin{aligned} M^*(T\zeta, T\kappa, T\kappa) &= \left|\frac{\zeta^2}{2} - \frac{\kappa^2}{2}\right| + \left|\frac{\kappa^2}{2} - \frac{\kappa^2}{2}\right| \\ &= \frac{1}{2}|\zeta^2 - \kappa^2| \\ &= \frac{1}{2}|\zeta + \kappa||\zeta - \kappa| \\ &\leq \frac{1}{2} \cdot 2 \cdot |\zeta - \kappa| \quad (since\ \zeta, \kappa \in [0,1]) \\ &= |\zeta - \kappa| \\ &\leq \frac{1}{2} \max\{|\zeta - \kappa| + 0, \quad (M^*(\zeta, \kappa, \kappa)) \end{aligned}$$

$$|\zeta - \frac{\zeta^2}{2}| + |\frac{\zeta^2}{2} - \frac{\zeta^2}{2}|, \quad (M^*(\zeta, T\zeta, T\zeta))$$

$$|\kappa - \frac{\kappa^2}{2}| + |\frac{\kappa^2}{2} - \frac{\kappa^2}{2}| \quad (M^*(\kappa, T\kappa, T\kappa))$$

3. ***Fixed Point Verification****:*

- *Solve* $T\zeta^* = \zeta^*$: $\frac{(\zeta^*)^2}{2} = \zeta^* \quad \zeta^* = 0$ *or* 2
- *Only* $\zeta^* = 0$ *is in* $X = [0, 1]$
- *Check uniqueness: No other solutions in* $[0, 1]$

4. ***Iterative Behavior****: For any* $\zeta_0 \in [0, 1]$*, the sequence:*

$$\zeta_{n+1} = \frac{\zeta_n^2}{2}$$

converges to 0 *since:*

$$\zeta_n \leq \left(\frac{1}{2}\right)^{2^n - 1} \zeta_0^{2^n} \to 0$$

3. Conclusion

This research has made substantial contributions to fixed point theory and its applications:

3.1 Theoretical Advancements

- Extended ´Ciri´c-type fixed point theorems to M∗-metric spaces
- Developed new contraction conditions building on [1, 4]

• Established connections with other generalized metric spaces [8, 10]

3.2 Practical Applications

• Demonstrated applications to nonlinear problems [21]

• Extended methods to fractional calculus [5, 23]

• Provided computational approaches using the atomic solution method [23]

3.3 Future Research Directions

• Extension to coupled fixed points in M∗-metric spaces

• Further applications in fractional differential equations [22]

• Development of computational algorithms [23]

• Investigation of new contraction types [3, 16]

These results establish a robust foundation for ongoing research in nonlinear analysis and its applications to various mathematical problems.

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