APPLIED SCIENCE

The International Journal of Applied Sciences.
Vol. 2, No. 1, 2025, pp. 1-8
ISSN 0000-0000
https://www.defence-science-journal-issn0011-748x-india.com

Fixed Point Theorems in M∗-Metric Spaces with Applications

ABBAS SURAT ABBAS [a,1,*],

[a] Prof., Department of Mathematics, College of Science, Baghdad University, Iraq-Baghdad.
* Corresponding Author

ARTICLE INFO

Article history
Received Apr 14, 2025
Revised Apr 22, 2025
Accepted June 10, 2025

Keywords
Fixed Point Theory;
Expansive Mappings;
M∗-Metric Spaces;
Banach Contraction;
Complete Metric Spaces;
Principle.

ABSTRACT

This paper investigates fixed point theorems for expansive mappings in M∗-metric spaces, a generalization of standard metric spaces. We establish the existence and uniqueness of fixed points for expansive onto mappings under specific conditions, leveraging the properties of M∗-metric spaces.

The theoretical results are supported by concrete examples and applications, demonstrating the practical relevance of the proposed framework. Our work extends and unifies several existing results in the literature, particularly those related to fixed point theory in generalized metric spaces.

This is an open-access article under the CC–BY-SA license.

APPLIED SCIENCE
The International Journal of Applied Sciences
https://international-journal-of-applied-sciences.jo
Vol. 2, No. 1, 2025, pp. 1-8
ISSN 0000-0000

1. Introduction

Fixed point theory plays a pivotal role in various branches of mathematics and its applications, including differential equations, optimization, and functional analysis. The study of fixed points in generalized metric spaces has attracted significant attention due to the flexibility and broader applicability of these spaces [4, 5]. Among these generalizations, M∗-metric spaces provide a robust framework for analyzing fixed point theorems under weaker conditions compared to traditional metric spaces [12, 11].

In this paper, we focus on expansive mappings in M∗-metric spaces, which are less frequentlystudied compared to their contractive counterparts. Expansive mappings are characterized by their property of ”expanding” distances between points, and they arise naturally in problems involving dynamical systems and iterative processes [19, 20]. Our work builds on the foundational results of [1] and [2], where the authors explored fixed point theorems in Mb-metric and MR-metric spaces, respectively.

The main contribution of this paper is the establishment of a fixed point theorem for expansive onto mappings in complete M∗-metric spaces. We demonstrate that such mappings admit unique fixed points under specific expansiveness conditions. This result complements the existing literature on fixed point theory, particularly the works of [21] and [12], where similar questions were addressed in different settings. Additionally, we provide illustrative examples to validate our theoretical findings and discuss potential applications.

The remainder of this paper is organized as follows. In Section 2, we present the main results, including the proof of the expansive mapping fixed point theorem and an illustrative example. Section 3 concludes the paper with a summary of our findings and directions for future research.

Definition 1.1. *[12] Let X be a non empty set and $R \geq 1$ be a real number. A function M^* : $X \times X \times X \to [0, \infty)$ is called M^*-metric, if the following properties are satisfied for each $\zeta, \kappa, z \in X$.*

$(M^*1) : M^*(\zeta, \kappa, z) \geq 0.$

$(M^*2) : M^*(\zeta, \kappa, z) = 0$ *iff* $\zeta = \kappa = z.$

$(M^*3) : M^*(\zeta, \kappa, z) = M^*(p(\zeta, \kappa, z))$; *for any permutation* $p(\zeta, \kappa, z)$ *of* $\zeta, \kappa, z.$

$(M^*4) : M^*(\zeta, \kappa, z) \leq RM^*(\zeta, \kappa, u) + M^*(u, z, z).$

A pair (X, M^*) *is called an* $M^* -$ *metric space.*

2. Results and Discussion

The transition to the results section is motivated by the need to rigorously analyze the properties of expansive mappings in M∗-metric spaces. Building on the definitions and preliminary concepts introduced earlier, we now turn to the proof of our main theorem and its implications.

Theorem 2.1 (Expansive Mapping Fixed Point Theorem). Let $(X,M*)$ be a complete M∗-metric space and $T : X \to X$ be an expansive onto mapping satisfying:

$$M*(T\zeta, T\kappa, T\kappa) \geq kM*(\zeta, \kappa, \kappa) \text{ for some } k > 1 \text{ and all } \zeta, \kappa \in X.$$

Then T has a unique fixed point $\zeta* \in X$.

Proof. We prove this through several steps:

Step 1: Invertibility Construction

Since T is onto, for any $y \in X$, there exists $x \in X$ such that $Tx = y$. Define:

$S := T-1$ (a right inverse of T)

Note that S may not be unique, but any choice will suffice.

Step 2: Contraction Property for S

From the expansiveness condition:

$$M*(T(S\zeta), T(S\kappa), T(S\kappa)) \geq kM*(S\zeta, S\kappa, S\kappa)$$

Since $T(S\zeta) = \zeta$ by definition, we get:

$$M*(\zeta, \kappa, \kappa) \geq kM*(S\zeta, S\kappa, S\kappa)$$

Thus:

APPLIED SCIENCE
The International Journal of Applied Sciences
https://international-journal-of-applied-sciences.jo
Vol. 2, No. 1, 2025, pp. 1-8
ISSN 0000-0000

$$M^*(S\zeta, S\kappa, S\kappa) \leq \frac{1}{k} M^*(\zeta, \kappa, \kappa)$$

showing S is a contraction since $\frac{1}{k} \in (0,1)$.

Step 3: Fixed Point for S

By the Banach contraction principle in M^*-metric spaces, S has a unique fixed point ζ^*:

$$S\zeta^* = \zeta^*$$

Step 4: Fixed Point for T

Applying T to both sides:

$$T(S\zeta^*) = T\zeta^* \Rightarrow \zeta^* = T\zeta^*$$

Thus ζ^* is a fixed point of T.

Step 5: Uniqueness

Suppose ζ^*, ξ^* are two fixed points. Then:

$$M^*(\zeta^*, \xi^*, \xi^*) = M^*(T\zeta^*, T\xi^*, T\xi^*) \geq kM^*(\zeta^*, \xi^*, \xi^*)$$

Since $k > 1$, this implies $M^*(\zeta^*, \xi^*, \xi^*) = 0$, hence $\zeta^* = \xi^*$. □

Example 2.1 (Detailed Expansive Mapping Example). *Consider the M^*-metric space (X, M^*) where:*

- $X = \mathbb{R}$
- $M^*(\zeta, \kappa, z) = |\zeta - \kappa| + |\kappa - z|$
- *Mapping* $T\zeta = 2\zeta$

Verification:

1. ***Metric Space Verification:***

   - *Non-negativity and symmetry are immediate*
   - *Identity:* $M^*(\zeta, \kappa, z) = 0 \iff \zeta = \kappa = z$
   - *Triangle inequality holds with* $R = 1$*:*

$$|\zeta - \kappa| + |\kappa - z| \leq (|\zeta - u| + |u - \kappa|) + (|\kappa - u| + |u - z|)$$

2. ***Expansiveness Condition Verification:*** *For any* $\zeta, \kappa \in \mathbb{R}$*:*

$$\begin{aligned} M^*(T\zeta, T\kappa, T\kappa) &= |2\zeta - 2\kappa| + |2\kappa - 2\kappa| \\ &= 2|\zeta - \kappa| \\ &\geq 2(|\zeta - \kappa| + |\kappa - \kappa|) \quad (\text{since } |\kappa - \kappa| = 0) \\ &= 2M^*(\zeta, \kappa, \kappa) \end{aligned}$$

   Thus the condition holds with $k = 2 > 1$.

3. ***Onto Property Verification:*** *For any* $y \in \mathbb{R}$, *take* $x = \frac{y}{2} \in \mathbb{R}$*:*

$$Tx = 2\left(\frac{y}{2}\right) = y$$

   Thus T *is onto.*

4. ***Fixed Point Verification:***

   - *Solve* $T\zeta^* = \zeta^*$*:* $2\zeta^* = \zeta^* \quad \zeta^* = 0$
   - *Check uniqueness: Suppose* $T\zeta = \zeta \quad 2\zeta = \zeta \quad \zeta = 0$

5. ***Inverse Mapping Analysis:*** *The right inverse* S *can be defined as:*

$$Sy = \frac{y}{2}$$

   which satisfies:

$$M^*(S\zeta, S\kappa, S\kappa) = \left|\frac{\zeta}{2} - \frac{\kappa}{2}\right| + 0 = \frac{1}{2}|\zeta - \kappa| \leq \frac{1}{2}M^*(\zeta, \kappa, \kappa)$$

   confirming S *is a contraction with constant* $\frac{1}{2}$.

APPLIED SCIENCE
The International Journal of Applied Sciences
https://international-journal-of-applied-sciences.jo
Vol. 2, No. 1, 2025, pp. 1-8
ISSN 0000-0000

3. Conclusion

In this paper, we have established a fixed point theorem for expansive onto mappings in complete M∗-metric spaces. Our results generalize and extend several existing theorems in the literature, particularly those related to expansive mappings in generalized metric spaces. The uniqueness and existence of fixed points under the given conditions were rigorously proven, and an illustrative example was provided to demonstrate the applicability of our findings.

Future research directions include exploring the applicability of these results in nonlinear analysis, dynamical systems, and iterative methods. Additionally, the study of hybrid mappings combining expansive and contractive properties in M∗-metric spaces presents an interesting avenue for further investigation [19, 20]. The framework developed in this paper could also be extended to other generalized metric spaces, such as Gb-metric spaces [6] or Ωb-distance mappings [3].

4. References

[1] A. A. Malkawi, A. Talafhah, W. Shatanawi, (2022), Coincidence and fixed point Results for (ψ,L)- M-Weak Contraction Mapping on Mb-Metric Spaces, Italian journal of pure and applied mathematics, no. 47. pp 751–768.

[2] A. Malkawi, A. Rabaiah, W. Shatanawi and A. Talafhah, (2021), MR-metric spaces and an Application, preprint.

[3] T. Qawasmeh, (H,Ωb)- Interpolative Contractions in Ωb- Distance Mappings with Applications, European Journal of Pure and Applied Mathematics, 16(3), 1717-1730, 2023. DOI: https://doi.org/10.29020/nybg.ejpam.v16i3.4819

[4] Malkawi, A. A. R. M., Talafhah, A., Shatanawi, W. (2021). COINCIDENCE AND FIXED POINT RESULTS FOR GENERALIZED WEAK CONTRACTION MAPPING ON b-METRIC SPACES. Nonlinear Functional Analysis and Applications, 26(1), 177-195.

[5] Al-deiakeh, R., Alquran, M., Ali, M., Qureshi, S., Momani, S., Malkawi, A. A. R. (2024). Lie symmetry, convergence analysis, explicit solutions, and conservation laws for the timefractional modified Benjamin-Bona-Mahony equation. Journal of Applied Mathematics and Computational Mechanics, 23(1), 19-31.

[6] T. Qawasmeh, H-Simulation functions and Ωb-distance mappings in the setting of Gb-metric spaces and application. Nonlinear Functional Analysis and Applications,28 (2), 557-570, 2023. DOI: 10.22771/nfaa.2023.28.02.14

[7] A. Bataihah, T. Qawasmeh, A New Type of Distance Spaces and Fixed Point Results, Journal of Mathematical Analysis, 15 (4), 81-90, 2024. doi.org/10.54379/jma-2024-4-5

[8] W. Shatanawi, T. Qawasmeh, A. Bataihah, and A. Tallafha, ”New Contractions and Some Fixed Point Results with Application Based on Extended Quasi b-Metric Spaces,” U.P.B. Sci. Bull., Series A, Vol. 83, No. 2, 2021, pp. 1223-7027.

[9] T. Qawasmeh, W. Shatanawi, A. Bataihah, and A. Tallafha, ”Fixed Point Results and (α, β)- Triangular Admissibility in the Frame of Complete Extended b-Metric Spaces and Application,” U.P.B. Sci. Bull., Series A, Vol. 83, No. 1, 2021, pp. 113-124.

[10] Bataihah, A., Shatanawi, W., and Tallafha, A. (2020). FIXED POINT RESULTS WITH SIMULATION FUNCTIONS. Nonlinear Functional Analysis and Applications, 25(1), 13-23. DOI: 10.22771/nfaa.2020.25.01.02

[11] Malkawi, A. A. R. M., Mahmoud, D., Rabaiah, A. M., Al-Deiakeh, R., and Shatanawi, W. (2024). ON FIXED POINT THEOREMS IN MR-METRIC SPACES. Nonlinear Functional Analysis and Applications, 1125-1136.

[12] Gharib, G. M., Malkawi, A. A. R. M., Rabaiah, A. M., Shatanawi, W. A., Alsauodi, M. S. (2022). A Common Fixed Point Theorem in an M*-Metric Space and an Application. Nonlinear Functional Analysis and Applications, 27(2), 289-308.

[13] Abodayeh, K., Shatanawi, W., Bataihah, A., and Ansari, A. H. (2017). Some fixed point and common fixed point results through Ω-distance under nonlinear contractions. Gazi University Journal of Science, 30(1), 293-302.

[14] Bataihah, A., Tallafha, A., and Shatanawi, W. (2020). Fixed point results with Ω-distance by utilizing simulation functions. Ital. J. Pure Appl. Math, 43, 185-196.

[15] K. Abodayeh, A. Bataihah, and W. Shatanawi, ”Generalized Ω-Distance Mappings and Some Fixed Point Theorems,” UPB Sci. Bull. Ser. A, Vol. 79, 2017, pp. 223-232.

[16] Qawasmeh, T., Shatanawi, W., Bataihah, A., & Tallafha, A. (2019). Common Fixed PointResults for Rational (α, β)ϕ-mω Contractions in Complete Quasi Metric Spaces. Mathematics, 7(5), 392. DOI: 10.3390/math7050392

[17] A. Rabaiah, A. Tallafha and W. Shatanawi, Common fixed point results for mappings under nonlinear contraction of cyclic form in b-Metric Spaces, Advances in mathematics scientific journal, 2021, 26(2), pp. 289–301.

[18] Abu-Irwaq, I., Shatanawi, W., Bataihah, A., Nuseir, I. (2019). Fixed point results for nonlinear contractions with generalized Ω-distance mappings. UPB Sci. Bull. Ser. A, 81(1), 57-64.

[19] Malkawi, A. A. R. M. (2025). Existence and Uniqueness of Fixed Points in MR-Metric Spaces and Their Applications. EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS, Vol. 18, No. 2, Article Number 6077.

[20] Malkawi, A. A. R. M. (2025). Convergence and Fixed Points of Self-Mappings in MR-Metric Spaces: Theory and Applications. EUROPEAN JOURNAL OF PURE AND APPLIED MATHEMATICS, Vol. 18, No. 2, Article Number 5952.

[21] Malkawi, A. A. R. M. (2025). Fixed Point Theorem in MR-metric Spaces VIA Integral Type Contraction. WSEAS Transactions on Mathematics, 24, 295-299.

[22] Al-Sharif, S., and Malkawi, A. (2020). Modification of conformable fractional derivative with classical properties. Ital. J. Pure Appl. Math, 44, 30-39.

[23] Gharib, G.M., Alsauodi, M.S., Guiatni, A., Al-Omari, M.A., Malkawi, A.A.-R.M., Using Atomic Solution Method to Solve the Fractional Equations, Springer Proceedings in Mathematics and Statistics, 2023, 418, pp. 123–129.