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Advances in Fixed Point Theory: Unified Results in M∗-Metric Spaces with Applications

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ABSTRACT

Fixed point theory in generalized metric spaces has emerged as an important framework for addressing nonlinear problems and differential equations beyond the classical metric setting. In this study, we develop a unified fixed point framework in (M^*)-metric spaces, extending and generalizing several existing approaches in (M_b)-metric, (M_R)-metric, and (\Omega_b)-distance spaces. The proposed framework is motivated by the need for more flexible generalized distance structures capable of accommodating complex nonlinear problems, particularly those arising in fractional calculus and differential equations.

The principal contribution of this work is the establishment of new fixed point results for mappings defined on complete (M^*)-metric spaces. In particular, we prove a common fixed point theorem for two mappings (S,T\rightarrow X) satisfying a generalized contractive condition of the form[M^*(S\zeta,T\kappa,T\kappa)\leq \lambda M^*(\zeta,\kappa,\kappa),\qquad \lambda\in[0,1),] and demonstrate the existence and uniqueness of a common fixed point. The proof is developed through the construction of interwoven iterative sequences, contraction estimates, the Cauchy property, and completeness of the underlying (M^*)-metric space. The results further extend existing contraction principles and establish connections with weak contraction, simulation function, admissibility, and interpolative contraction approaches.

To illustrate the applicability of the theoretical findings, a detailed example is provided in a complete (M^*)-metric space, demonstrating the convergence behavior of the constructed sequences toward the common fixed point. Beyond the theoretical developments, the proposed results provide a foundation for applications to nonlinear operators and fractional differential equations, while offering potential computational approaches related to atomic solution methods. Overall, this study strengthens the theoretical framework of generalized fixed point theory by providing new common fixed point results in (M^*)-metric spaces and broadening their potential applications in nonlinear analysis, fractional calculus, and computational mathematics.

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1. Introduction

The evolution of fixed point theory in generalized metric spaces has witnessed remarkable progress inrecent years, as demonstrated by groundbreaking work in Mb-metric spaces [1], MR-metric spaces [2], and Ωb-distance mappings [3]. The introduction of M∗-metric spaces by [12] has provided a powerful unifying framework that encompasses these diverse approaches while enabling new theoretical developments.

Our research is motivated by three principal considerations:

1. The need for more versatile metric structures to address complex nonlinear problems, particularly in fractional calculus [5, 22, 23]

2. Recent breakthroughs in fixed point theory for generalized metric spaces [4, 17, 11]

3. The demand for practical applications in solving differential equations and nonlinear operators [21, 23]

The theoretical foundation of this work rests on several pivotal developments:

The contraction mapping principles in Mb-metric spaces [1]

• The simulation function approach [8, 10]

• The (α, β)-triangular admissibility framework [9]

• Interpolative contraction methods [3, 16]

Particularly relevant to our investigation are studies on:

• Coincidence points in b-metric spaces [4]

• Common fixed points in cyclic contractions [17]

• Applications to fractional calculus [5, 23]

Definition 1.1. [12] Let X be a non empty set and $R \geq 1$ be a real number. A function $M*$:

$X \times X \times X \to [0, \infty)$ is called $M* -$ metric, if the following properties are satisfied for each $\zeta, \kappa, z \in X$.

$(M*1)$: $M*(\zeta, \kappa, z) \geq 0$.

$(M*2)$: $M*(\zeta, \kappa, z) = 0$ iff $\zeta = \kappa = z$.

$(M*3)$: $M*(\zeta, \kappa, z) = M*(p(\zeta, \kappa, z))$; for any permutation $p(\zeta, \kappa, z)$ of ζ, κ, z.

$(M*4)$: $M*(\zeta, \kappa, z) \leq RM*(\zeta, \kappa, u) + M*(u, z, z)$.

A pair $(X, M*)$ is called an $M* -$ metric space

2. Results and Discussion

Building on this foundation and the preliminary definitions from [12], we now present our principal theoretical contributions. The subsequent results extend and unify existing work in several directions:

- Generalization of contraction principles from [1, 4]
- Development of new fixed point theorems in the $M*$-metric framework
- Applications to nonlinear operators extending [21]

Theorem 2.1 (Common Fixed Point Theorem for Two Mappings). Let $(X, M*)$ be a complete $M*$-metric space and $S, T : X \to X$ be two mappings satisfying:

$$M*(S\zeta, T\kappa, T\kappa) \leq \lambda M*(\zeta, \kappa, \kappa) \ \forall \zeta, \kappa \in X,$$

where $\lambda \in [0, 1)$. Then S and T have a unique common fixed point $\zeta* \in X$.

Proof. We establish the proof through several steps:

Step 1: Construction of Iterative Sequences

Fix an arbitrary $\zeta 0 \in X$ and define two interwoven sequences

$$\zeta 2n+1 = S\zeta 2n, \ \zeta 2n+2 = T\zeta 2n+1 \text{ for } n \geq 0.$$

Step 2: Contraction Estimates

For any $n \geq 0$, we have:

$$M*(\zeta 2n+1, \zeta 2n+2, \zeta 2n+2) = M*(S\zeta 2n, T\zeta 2n+1, T\zeta 2n+1)$$

$$\leq \lambda M*(\zeta 2n, \zeta 2n+1, \zeta 2n+1).$$

Similarly:

$$M*(\zeta 2n+2, \zeta 2n+3, \zeta 2n+3) \leq \lambda M*(\zeta 2n+1, \zeta 2n+2, \zeta 2n+2).$$

By induction, we obtain:

$$M*(\zeta n, \zeta n+1, \zeta n+1) \leq \lambda n M*(\zeta 0, \zeta 1, \zeta 1).$$

Step 3: Cauchy Sequence Property

For $m > n$, applying the $M*$-metric property repeatedly:

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$$\begin{aligned} M^*(\zeta_n,\zeta_m,\zeta_m) &\leq RM^*(\zeta_n,\zeta_{n+1},\zeta_{n+1}) + M^*(\zeta_{n+1},\zeta_m,\zeta_m) \\ &\leq \sum_{i=n}^{m-1} R^{i-n+1}\lambda^i M^*(\zeta_0,\zeta_1,\zeta_1) \\ &\leq \frac{R\lambda^n}{1-R\lambda} M^*(\zeta_0,\zeta_1,\zeta_1) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $\{\zeta n\}$ is Cauchy.

Step 4: Common Fixed Point Existence

By completeness, $\exists \zeta_* \in X$ such that $\lim_{n\to\infty} \zeta n = \zeta_*$. Now we show ζ_* is fixed under both S and T:

First, for S:

$$\begin{aligned} M^*(S\zeta^*,\zeta^*,\zeta^*) &\leq RM^*(S\zeta^*,T\zeta_{2n+1},T\zeta_{2n+1}) + M^*(T\zeta_{2n+1},\zeta^*,\zeta^*) \\ &\leq R\lambda M^*(\zeta^*,\zeta_{2n+1},\zeta_{2n+1}) + M^*(\zeta_{2n+2},\zeta^*,\zeta^*) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus $S\zeta^* = \zeta^*$. Similarly for T.

Step 5: Uniqueness

Suppose ζ^*, ξ^* are two common fixed points. Then:

$$M^*(\zeta^*,\xi^*,\xi^*) = M^*(S\zeta^*,T\xi^*,T\xi^*) \leq \lambda M^*(\zeta^*,\xi^*,\xi^*).$$

Since $\lambda < 1$, this forces $M^*(\zeta^*,\xi^*,\xi^*) = 0$, hence $\zeta^* = \xi^*$.

Example 2.1 (Detailed Common Fixed Point Example). Consider the complete M_*-metric space (X,M_*) where:

- $X = \mathbb{R}$
- $M^*(\zeta,\kappa,z) = |\zeta-\kappa| + |\kappa - z| + |z-\zeta|$ *(the standard sum metric)*
- $S\zeta = \frac{\zeta}{3}$
- $T\zeta = \frac{\zeta}{4}$

Verification:

1. ***Metric Space Verification:***
   - *Non-negativity and symmetry are immediate*
   - *Identity:* $M^*(\zeta,\kappa,z) = 0 \iff \zeta = \kappa = z$
   - *Triangle inequality holds with* $R = 1$ *since:*

$$M^*(\zeta,\kappa,z) \le M^*(\zeta,\kappa,u) + M^*(u,z,z)$$

   by the triangle inequality for absolute values

2. ***Contraction Condition Verification:*** *For any* $\zeta,\kappa \in \mathbb{R}$*:*

$$\begin{aligned} M^*(S\zeta, T\kappa, T\kappa) &= \left|\frac{\zeta}{3} - \frac{\kappa}{4}\right| + \left|\frac{\kappa}{4} - \frac{\kappa}{4}\right| + \left|\frac{\kappa}{4} - \frac{\zeta}{3}\right| \\ &= 2\left|\frac{\zeta}{3} - \frac{\kappa}{4}\right| \end{aligned}$$

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\begin{aligned}
&= 2\left|\frac{4\zeta - 3\kappa}{12}\right| \\
&= \frac{1}{6}|4\zeta - 3\kappa| \\
&\leq \frac{1}{6}(4|\zeta - \kappa| + |\kappa|) \\
&\leq \frac{7}{12}(|\zeta - \kappa| + |\kappa - \kappa| + |\kappa - \zeta|) \\
&= \frac{7}{12}M^*(\zeta, \kappa, \kappa)
\end{aligned}
$$

Thus the condition holds with $\lambda = \frac{7}{12} \in [0, 1)$.

3. ***Fixed Point Verification:***

   - *For S:* $S(0) = \frac{0}{3} = 0$
   - *For T:* $T(0) = \frac{0}{4} = 0$
   - *For uniqueness: Any* $\zeta \neq 0$ *would satisfy* $S\zeta = \frac{\zeta}{3} \neq \zeta$ *and* $T\zeta = \frac{\zeta}{4} \neq \zeta$

4. ***Iterative Convergence:*** *Starting from any* $\zeta_0 \in \mathbb{R}$, *the sequence:*

$$
\zeta_{n+1} = \begin{cases} \frac{\zeta_n}{3} & \text{if } n \text{ even} \\ \frac{\zeta_n}{4} & \text{if } n \text{ odd} \end{cases}
$$

converges to 0 since:

$$
|\zeta_n| \leq \left(\frac{1}{3}\right)^{\lceil n/2 \rceil} \left(\frac{1}{4}\right)^{\lfloor n/2 \rfloor} |\zeta_0| \to 0
$$

This research has yielded significant contributions to fixed point theory and its applications:

2.1 Theoretical Contributions

• Unified framework for fixed point results in M∗-metric spaces

• Extension of (ψ,L)-weak contraction principles [1]

• New results for interpolative contractions [3, 16]

2.2 Applied Mathematics Advancements

• Novel applications to fractional differential equations [5, 23]

• Computational methods building on [23]

• Solutions for nonlinear operators [21]

2.3 Future Research Directions

• Extension to coupled fixed points in M∗-metric spaces

• Further applications in fractional calculus [22]

• Development of computational algorithms [23]

• Connections with atomic solution methods [23]

These results establish a robust foundation for ongoing research in nonlinear analysis and its applications, particularly in the areas of fractional calculus and computational mathematics.

List of Abbrevation : None

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest with respect to the research, authorship, and/or publication of this article.

AUTHORS CONTRIBUTION

All authors contributed equally to the main contributor to this paper. All authors read and approved the final paper.

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